Traffic Jams

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Abstract

The flow of vehicular traffic has been studied in many different ways. After an introduction, we will give overview of several models for the flow on single lane road: a fluid dynamics theory, a approach based on the dynamics of individual vehicles and a model using a cellular automata. The latter model will be used to study to flow in two dimensions representing the traffic in a city with crossings and traffic lights.

1 Introduction

The problems of transportation and traffic are as old as man's history of transportation. The type of vehicles changed over the centuries and so did traffic problems and the ways one tried to solve them. The evolution of traffic in this century can be characterized by a vast increase in the number of cars. This process was followed by an expanding network of roads and streets trying to keep up with the increasing demand. At the same time new methods to control the traffic were advanced. They include simple and intelligent traffic lights as well as sufficient information and rerouting services.

Essential for the development of such a controlling mechanism is a detailed understanding of the traffic flow. A large number of models have been applied. Most of them are concerned with the one-dimensional vehicle flow on a road without intersections. In this paper, three different approaches will be described: a fluid dynamics model, a model based on the dynamics of individual vehicles and cellular automata (CA). All three predict traffic jams when a critical density is reached but the properties of the congested phases are rather different. The traffic flow in cities requires two-dimensional models. Three different approaches based on CA will be discussed.

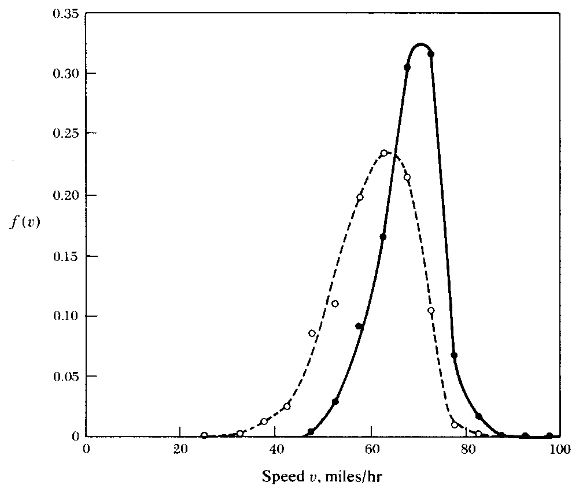


Fig. 1: Measured velocity distribution function f(v) versus velocity v. (taken from [1]).

2 Basic Concepts of Traffic Flow and Kinetic Theory

An overview of the field is given in the work by Prigogine *el al.* [1] on kinetic theory of vehicular traffic. The traffic flow can be understood by studying the distribution function $f(x, v, t)$, which describes the number of cars in the road interval $(x, x+dx)$ with a velocity in $(v, v+dv)$ at a given time t. A few measurements have been made

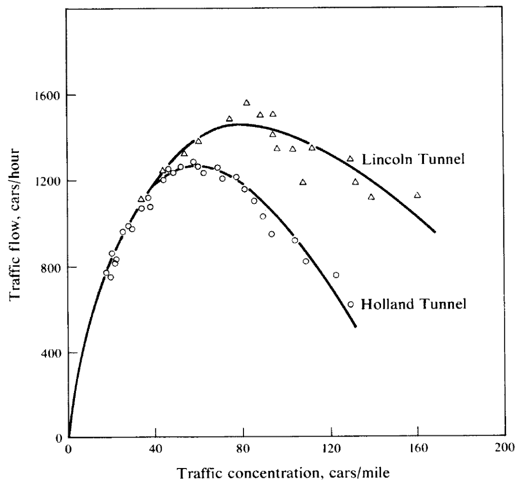


Fig. 2: Fundamental diagram: Flow versus vehicle concentration is shown in this *fundamental diagram* for two different tunnels. (taken from [1]).

to determine this function (see Fig. 1). This function can also be expressed in terms of the two *fundamental* variables, *concentration* and *flow* (or current). An understanding of the relation of concentration and flow is the basis for describing the traffic. The flow-concentration curve is called the *fundamental diagram* of road traffic or the *equation of state* of traffic theory. Two measurements of this function are shown in Fig. 2. At low densities, the flow increases linearly with concentration. Cars move independently in the dilute traffic with small fluctuations about a mean velocity. With increasing concentration, cars hinder each other more and more, which leads to a reduction of the average velocity and a decreasing slope in the fundamental diagram. These hindering effects become dominant at high concentrations. At a *critical density*, the flow exhibits a maximum. Then, an increase in density results into an actual decrease of the total throughput, and finally in a completely congested phase where no car can move and the flow is zero.

The flow near the critical concentration is unstable. Small perturbations that increase the density locally lead to a decrease in the flow, which then amplifies the initial perturbation. The *variance* in car velocities is what triggers this process [6], which leads to the formation of traffic jams. Traffic jams are locally congested phases, in which cars travel at slow or zero velocity. Between jams, one finds cars at low concentrations traveling with high velocities. These congested phases are rather stable configurations with their own dynamics. Cars driving into the jams stabilize and cars leaving from the front can reduce the length. From this picture, it follows that jams move slowly in the opposite direction of the cars. Their velocity can be estimated in a very simple model where one assumes a small constant velocity v_1 and a density ρ_1 in the jam and a higher velocity v_2 and low density ρ_2 outside. If cars accelerate and decelerate instantly, the jam moves with a velocity,

$$v_{jam} = -(v_2 - v_1)\frac{\rho_1}{\rho_2} + v_1 \quad . \tag{1}$$

Since the density ratio is small, jams move slowly. A much more sophisticated picture of jams is given by the theory of fluid dynamics, in which jams are modeled in terms of density waves.

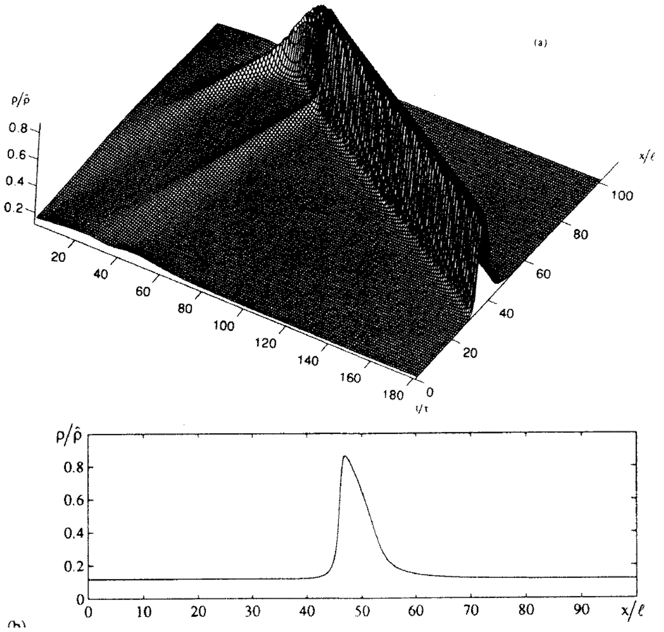


Fig. 3: Time evolution of a traffic jam showing the density $\rho(x,t)$ upper graph and $\rho(x)$ for a particular time (taken from [3]).

3 Fluid Dynamics

Fluid-dynamic approaches have been developed since the 1950's [2]. The road traffic is modeled as a one-dimensional compressible flow with density $\rho(x,t)$ with velocity $v(x,t)$. The continuity equation reads,

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v)}{\partial x} = 0 \tag{2}$$

The dynamics can be expressed by the Navier-Stokes equation [3], which states that the particle density times acceleration equals the sum of the acting forces,

$$\rho\left(\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x}\right) = \frac{\partial}{\partial t}\left(\mu\frac{\partial v}{\partial x}\right) - \frac{\partial p}{\partial x} + X \quad , \tag{3}$$

where p is the local pressure of the vehicular gas, μ is the viscosity and X represents the force caused by the interaction of the individual vehicles. The forces are derived from the density dependence of the velocity given by the fundamental diagram such as shown in Fig. 2,

$$X = \rho(v(\rho) - v)/\tau, \tag{4}$$

where τ is the relaxation time of the process. Kerner *el al.* [3, 4, 5] showed that this ansatz leads to the formation of stationary clusters (jams). Their form is shown in Fig. 3. The model was significantly improved by Helbing [6] who introduced the *variance* of the velocities as an additional space and time dependent variable. He showed the this approach leads to a more realistic behavior when jams develop.

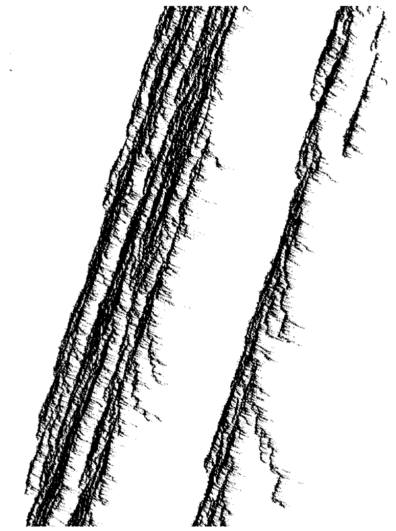


Fig. 4: Space-time plot of a traffic jam in the cellular automata model by Nagel *el al.* [7]. The horizontal direction is space and the vertical time. Only the position of cars in jams are shown.

4 Cellular Automata

Cellular Automata have the advantage of modeling the traffic flow on the microscopic scale of individual vehicles and allow the study of large systems due to a simple type of dynamics. A one-dimensional model with periodic boundary conditions representing a single lane freeway was introduced by Nagel *el al.* [8, 9] and later slightly modified in [7]. The sites of the array of length L can be occupied by a vehicle with velocity between zero and v_{max} or be empty. The system is updated in each time step performing the following steps:

1. *Free travel:* A car traveling with v_{max} and sufficient free space ahead keeps this velocity and the two following steps are skipped,

2. *Acceleration:* If the distance to the next car is big enough and $v < v_{max}$, v is increased by 1 with a probability of $\frac{1}{2}$,

3. *Slowing down:* If there is another car j sites ahead the velocity is reduced to $j-1$. With probability of $\frac{1}{2}$, it overacts and slows down even further to $j-2$,

4. *Movement:* Each vehicle advances v sites.

This model has been studied intensively using parallel computers, in particular, the lifetime of emerging traffic jams such as shown in Fig. 4. It was shown that the probability distribution as a function of lifetime follows a power law,

$$P(t) \sim t^{(\delta+1)} \quad \text{with} \quad (\delta+1) = 1.5 \pm 0.01. \tag{5}$$

This exponent is very close to $\frac{3}{2}$, which is the first return-time exponent for a one-dimensional random walk. This hypothesis was pursued by studying the probability distribution $P(n,t)$ of a jam of size n at time t analytically. The time-dependence of the this distribution is given by,

$$P(n,t+1) = (1 - r_{in} - r_{out})\, P(n,t) + r_{in}\, P(n-1,t) + r_{out}\, P(n+1,t). \tag{6}$$

This equation describes the growing and shrinking of a jam, where r_{in} and r_{out} are density dependent parameters for the rate of cars driving in and out of the jam. From this equation, the analogy to a one-dimensional random walk becomes clear.

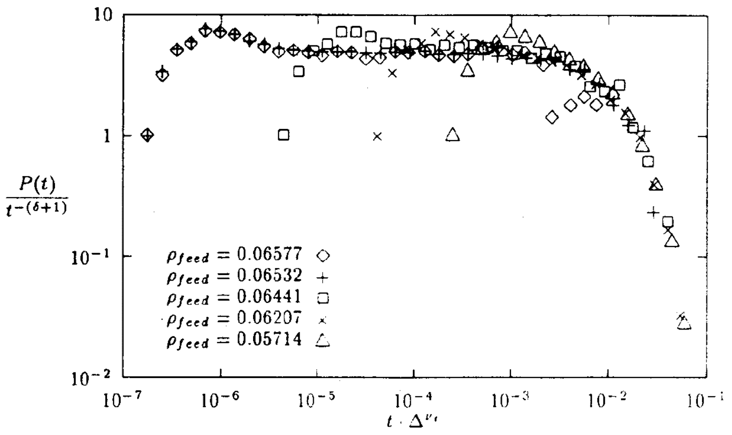


Fig. 5: Data collapse for the lifetime distribution of jams in the cellular automata model by Nagel *el al.* [7].

A jam starts at size $n = 1$ at particular time. It grows, then shrinks and finally disappears if n is equal to one at a later time. This is the reason why the life time of a jam can be related to first return-time of one-dimensional random walk. Going to the continuum limit in Eq. 6, it can be shown that P follows a scaling function,

$$P(t,\Delta) \sim t^{-(\delta+1)} f(t\Delta^{\nu_t}) \tag{7}$$

near the critical point with critical density ρ_c. Δ is defined as $\Delta = |\rho - \rho_c|$. In the case of $v_{max} = 1$, it follows exactly $\nu_t = 2$ and $\delta = \frac{1}{2}$. For the numerical results for $v_{max} = 5$, a data collapse was performed using $1 + \delta = 1.5$ (Fig. 5). It was found,

$$\nu_t = 2 \pm 0.2 \quad , \tag{8}$$

which agrees with prediction from the random walk model.

5 Dynamic Model

An interesting new approach was proposed by Bando *el al.* [10] who modeled the traffic flow in one dimension by solving an equation of motion for each individual car,

$$\ddot{x}_n = a\left\{V(\Delta x_n) - \dot{x}_n\right\} \quad , \quad \Delta x_n = x_{n+1} - x_n. \tag{9}$$

x_n denotes the position of vehicle number n ($1 \leq n \leq N$). This is a special case of the follow-the-leader theory (see [1]). It is assumed that there exists a legal velocity $V(\Delta x)$, which depends on the distance to the preceding vehicle. If this distance becomes too small the velocity must be reduced in order to prevent a crash. On the other hand, if the distance increases the vehicle can accelerate until it reaches a given speed limit. Thus, $V(\Delta x)$ has to be a monotonically increasing function, which satisfies $V(0) = 0$ and $V(\Delta x \to \infty) = V^{max}$.

$V(\Delta x)$ is given by the fundamental diagram. Its functional form plays a crucial role in the dynamics of the traffic flow. This can be understood by studying the stability of the steady state solution of of Eq. 9,

$$x_n^0 = s\,n + v\,t \qquad \text{with} \qquad s = L/N\ ,\ v = V(s) \tag{10}$$

A stability analysis shows that this solution is only stable if

$$V'(s) < \frac{a}{2}. \tag{11}$$

In [10], it has been suggested to choose,

$$V(\Delta x) = \tanh(\Delta x - 2) + \tanh 2, \tag{12}$$

and set $a = 1$. This function leads to a realistic behavior of the traffic flow. The steady state solution is unstable for $1.12 < s < 2.88$ and stable otherwise. This leads to two stable flow regimes, a low densities phase with fast moving vehicles in large separations and a congested phase with slowly moving cars at short distances. Both phases can be identified in Fig. 6 showing the evolving flow. The steady state solution in unstable regime with a small perturbation was chosen as the initial condition. At the beginning, several small jams form, which later merge to bigger congested phases. These move slowly in opposite direction to the cars, which travel at a high velocity from jam to jam. This scenario represents a stop-and-go behavior.

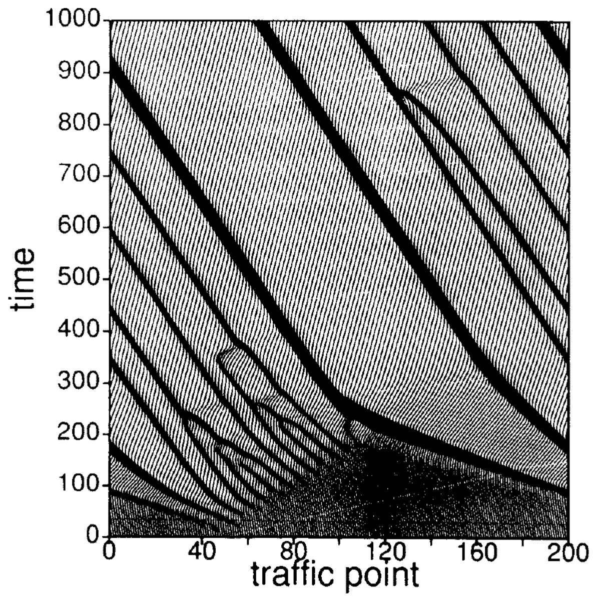


Fig. 6: The formation of traffic jams modeled by Bando *el al.* [10]. The position of the vehicles are shown as function of time. The initially uncongested flow soon develops small jams, which merge later.

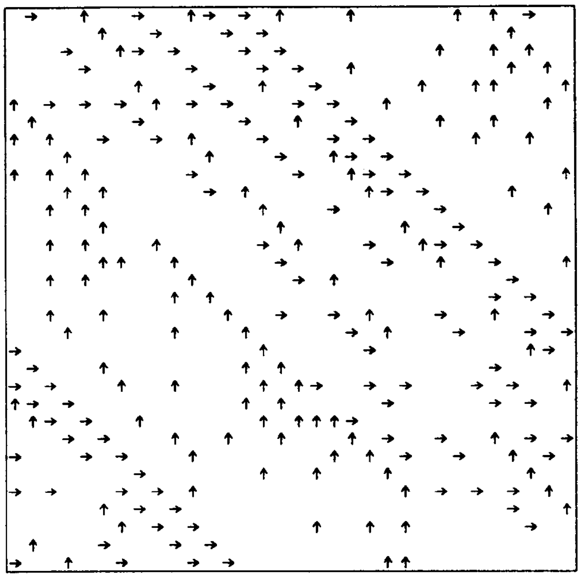
6 Two-dimensional models

In previous sections, we studied the traffic flow in one dimension. However, in cities the problem is much more complex. One has to consider the effect of intersection, traffic lights, coupling of traffic in different directions, turning cars, etc. Several two-dimensional models were advanced, which allows to incorporate those effects to a different extent.

A simple CA model in two dimensions was developed by Biham *el al.* [11]. The model is defined on a square lattice with periodic boundary conditions. Each site can be in three states, it contains a car moving upwards, a car moving to the right or it is empty. The dynamics is controlled by traffic lights so that cars can move to the right at odd time steps and upwards on the even time steps. In one time step, cars can only move to the next site if this is empty.

The described system exhibits two different regimes. At low densities, one finds a dynamic phase, where the system self-organizes similar cars along the diagonal and therefore maximizes the average speed (see Fig. 7). At high densities, all cars are jammed in one big cluster, which leads to an average velocity of zero. The two phases are separated by a sharp transition at a critical density, which is size dependent.

In the model described above, cars can never turn. This possibility was included in a more advanced model by Cuesta *el al.* [13, 14]. There, a parameter γ is



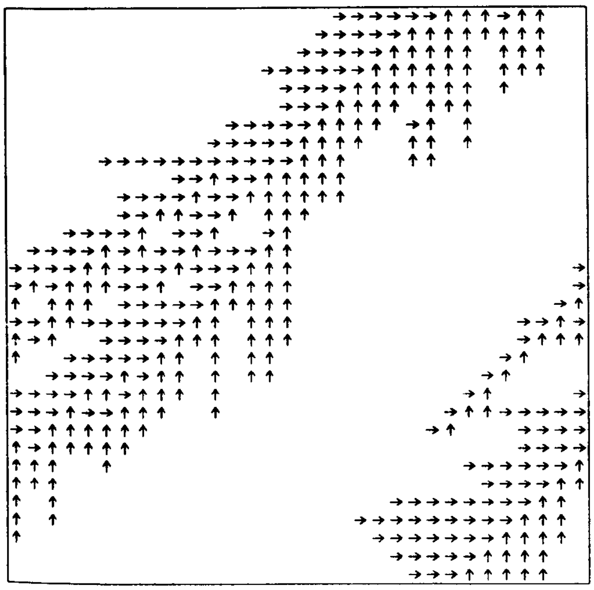


Fig. 7: Two configurations of in Bihams two-dimensional cellular automata model by Biham *el al.* [11], a jam-free phase in the upper figure and congested phases below. The arrows denote the direction of moving cars.

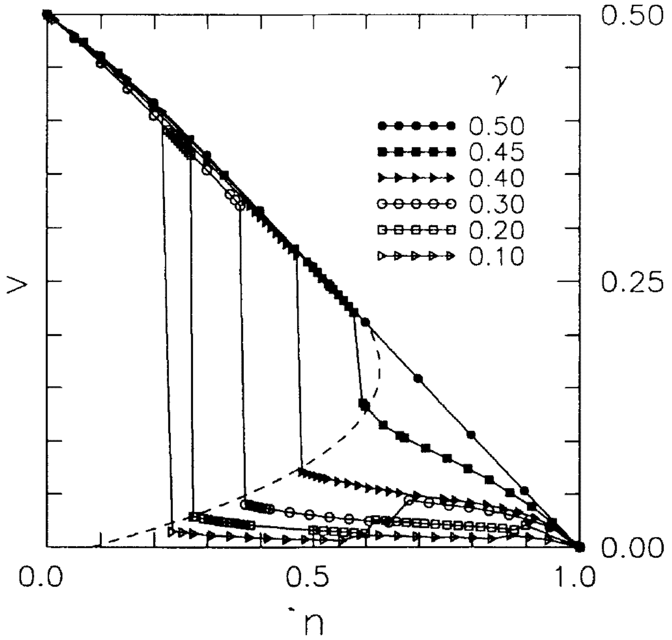


Fig. 8: Average velocity versus car density at different randomness value γ by Cuesta *el al.* [13]

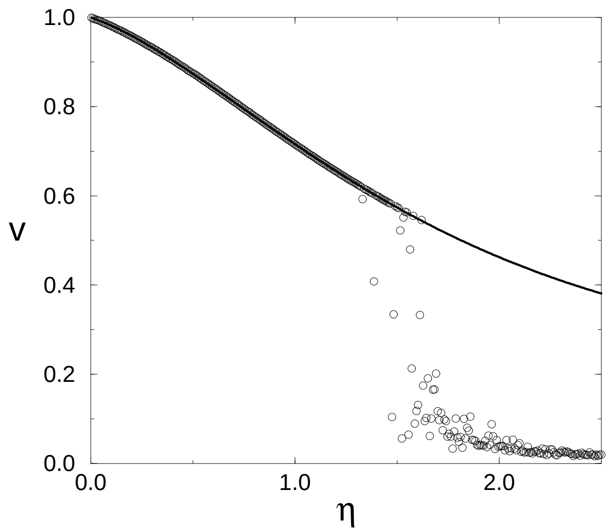


Fig. 9: Average velocity versus density from simulation (○) and statistical methods (solid line) by Freund *el al.* [12]

introduced, which determines the direction cars try to move. For one half of the cars, γ specifies the probability of trying to move upwards and $1-\gamma$ the probability of trying to move to the right. For the other half of the cars, the opposite probabilities are assigned. Setting $\gamma = 0$, this reproduces Biham's model. An increased γ gives car the possibility to take advantage of open sites out of their main direction of motion. This flexibility shifts the jamming transition to a higher density. This is shown in Fig. 8, which can be explained in terms of a first order transition critical point between $0.45 \leq \gamma \leq 0.50$.

In a following paper [14], the same system was studied using a non-linear Boltzmann equation. This approach can predict the existence of the jamming transition and physical quantities but does not show the correct γ dependence.

A much more elaborate model with *realistic traffic rules* was given by Freund *el al.* [12]. In this model, a city is represented by L vertical and horizontal two-way streets, which leads to L^2 crossing. At each crossing, cars can line up forming queues in every direction with a maximum length Q. There are no traffic lights but cars follow the left-before-right traffic rule, which is standard in many countries (for details see [12]). Each time a car lines up at a crossing the desired direction is chosen at random. A car can only cross an intersection if it has the right of way and the desired street is not fully occupied.

Numerical studies in [12] show a sharp jamming transition near a critical car density of $N/L^2 \approx 1.6$ (Fig. 9). Above this density, one still finds a remanent average velocity, which indicates that some jams may dissolve under those rules.

The numerical results were compared to predictions from statistical theory, in which the two-dimensional flow was described by a Bernoulli-process. The agreement is remarkably good until the jamming transition occurs (see Fig. 9).

7 Conclusions

The one and two-dimensional model discussed in this paper give insight in the principle nature of traffic jams on roads and in cities. In most models, the traffic flow is significantly simplified, which allowed analytical and numerical modeling. However, properties of individual drivers and cars as well as road conditions were not taken into account. Considering these factors will lead to more realistic models for the traffic flow. On the other hand, only a few comparisons with experimental data have been made. Experimental finding like reported by Kerner *el al.* [15] will certainly have an influence on the modeling and understanding of traffic flow.

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